Fundamentals

Some fundamental properties and initial concepts related to line-to-line functions will be developed in this chapter. These will provide the foundation for introducing concepts concerning direction later.

3.1 Derivative Comparisons

Consider the point-to-line function that was previously discussed. By virtue of the

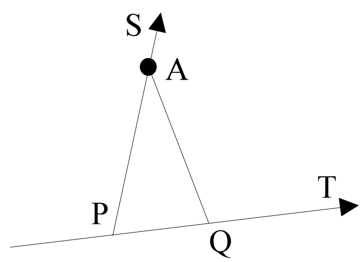


Figure 5: Distance from a Point to a Line

continuity axiom the distance *d(A,Q)* will be some differentiable function of *d(P,Q)* for *Q* to the right of *P*. Does the limit of the derivative of this function as *d(P,Q)* tends toward zero have some geometrical significance characteristic of the intersection at *P*?

Part of the answer to this question can be obtained by using the geodesic hypothesis to relate the derivatives of adjacent point-to-line functions. Compare a pair of points,

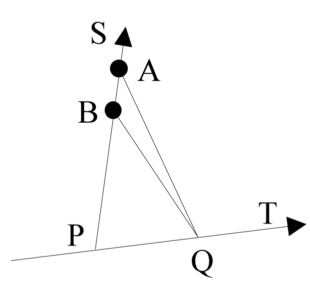


Figure 6: Collinear Points on One Side of a Line

A and *B*, on the same side of a line which are collinear with a point, *P*, on that line. As in the figure above take *B* between *A* and *P*. Both the distances from these points to a point, Q, moving about on the line in the vicinity to the right of P will, as before, be differentiable functions of *d(P,Q)*. In this situation: (1) the triangle inequality implies that for the small triangle $d(A,Q) \le d(A,B) + d(B,Q)$ and (2) the following lemma can be used.

Lemma. If *f* and *g* are two real valued, n-times ($n \ge 2$) differentiable functions such that $f(0) = g(0)$ and otherwise $f(x) < g(x)$ then, in particular,

$$\text{(i)}\quad f'(0) = g'(0) \quad \text{and} \quad \text{(ii)}\quad f''(0) \le g''(0)$$

and, in general, (iii) either $f^{(m)}(0) = g^{(m)}(0)$ for all $0 \le m \le n$ or there is an even $k \le n$ such that $f^{(m)}(0) = g^{(m)}(0)$ for $0 \le m < k$ and $f^{(k)}(0) < g^{(k)}(0)$ strictly.

Proof. If $h(x) = g(x) - f(x)$ then, according to the hypothesis, this function has an absolute minimum at zero. Consequently*, $h'(0) = 0$ and $h''(0) \ge 0$, which are the particular conclusions, and the general conclusions follow similarly.

Take $f = d(A,Q) - d(A,P)$ and $g = d(B,Q) - d(B,P)$ as a function of $\overline{PQ}$. Then, in view of the fact that $d(P,A) = d(P,B) + d(B,A)$, the lemma applies.

$$\left[\frac{d}{d\overline{PQ}} d(A,Q)\right]_{\overline{PQ}=0} = \left[\frac{d}{d\overline{PQ}} d(B,Q)\right]_{\overline{PQ}=0}$$

$$\left[\frac{d^2}{d\overline{PQ}^2} d(A,Q)\right]_{\overline{PQ}=0} \le \left[\frac{d^2}{d\overline{PQ}^2} d(B,Q)\right]_{\overline{PQ}=0}$$

Now $\overline{PQ}$ coincides with *d(P,Q)* for *Q* to the right of *P*. Therefore the above must be the limit of the derivatives with respect to *d(P,Q)* as it approaches zero. That is, the

* Apostol. Theorem 7-7 (p148).

first derivative is *independent* of the point on the intersecting line from which the test segment is dropped. Then the burden of the inequality falls on the second derivative which must increase as the intersection is approached.

Similarly, if a segment AB intersects a line at a point, P, then the same sort of argument can be used. Since AB is a segment of a line then by addition and the

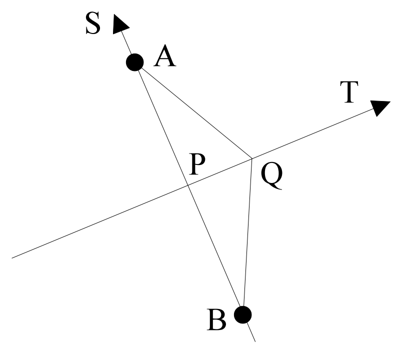


Figure 7: Collinear Points on Opposite Sides of an Intersecting Line

triangle inequality: $d(A,P)+d(P,B)=d(A,B)\le d(A,Q)+d(Q,B)$. Applying the lemma to $f=-\left(d(A,Q)-d(A,P)\right)$ and $g=d(B,Q)-d(B,P)$ yields

$$\left[-\frac{d}{d\overline{PQ}}d(A,Q)\right]_{\overline{PQ}=0}=\left[\frac{d}{d\overline{PQ}}d(B,Q)\right]_{\overline{PQ}=0}$$

and $$\left[-\frac{d^2}{d\overline{PQ}^2}d(A,Q)\right]_{\overline{PQ}=0}\le\left[\frac{d^2}{d\overline{PQ}^2}d(B,Q)\right]_{\overline{PQ}=0}.$$

The first derivatives on opposite sides of the intersecting line must, therefore, be equal in magnitude and opposite in sign.

The second derivative is constrained by the inequality but, at this point, there does not seem to be any more conclusive result. It will be eventually be possible to show, however, that infinitesimally close to the point of intersection it must be much like in Euclidean geometry: large, tending to the reciprocal of the distance, and positive. In general, still remaining is a strange possibility that sufficiently far away from the line the second derivative on one side might actually go negative forcing the second derivative on the other side to always remain above some finite positive value.

The sign of the first derivative is reversed if the point is moved across the line. It is also reversed for Q to the left of P since then, in repeating the arguments, the relationship between $\overline{PQ}$ and $d(P,Q)$ is opposite. Thus, as represented schematically

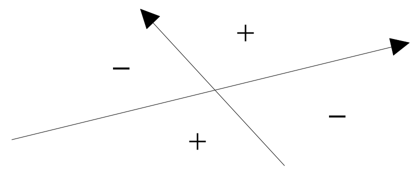


Figure 8: "Supplementary and Vertical Angles"

in the figure, this quantity behaves rather like supplementary and vertical angles. This suggests that there is really only one quantity and that the derivatives for the various configurations could, and should, be consolidated as a single concept.

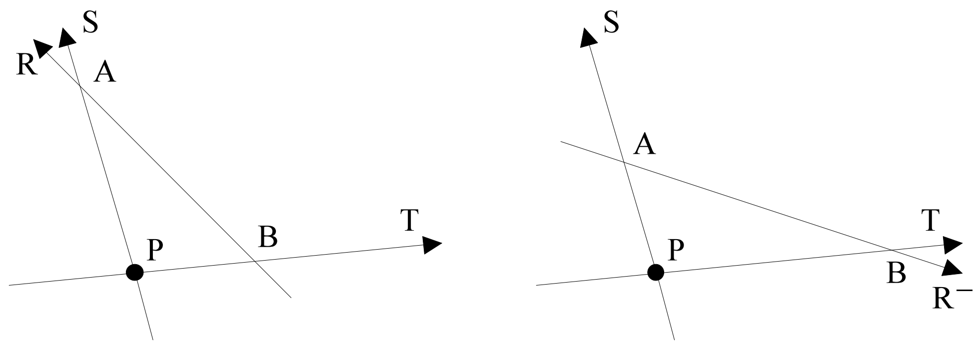
It is apparent that this is accomplished if the point-to-line function were altered to have the opposite sign on opposite sides of the line and if the signed measure were used as the variable. That is, the concepts are unified by using the signed segment measure, both for the value of the function and for its argument, rather than the distance function. The resulting function is just the line-to-line function which was previously defined. Its derivative at zero is then a quantity which is characteristic of the intersection and determined only by the two crossing rays, provided their orientation is properly taken into account. This is how the need to consider orientation and use the signed measure arises because it is useful and natural. And this motivates the closer examination of the properties of the line-to-line functions.

3.2 Line-to-Line Functions

There are two different line-to-line functions determined by two directions. The defining situations, as illustrated in the figure below, are basically the same. The segment measures that are the values of the functions ($\overline{BA}$ and $\overline{AB}$, respectively) are, compared to each other, of opposite order. But the other factor in evaluating these measures is the orientation of their respective lines, which may be different. While these crossbars are of opposite orientation in the first and third "quadrants" they have the same orientation in the second and fourth. So,

$$S_T(\overline{PA};\overline{PB}) = \pm T_S(\overline{PB};\overline{PA})$$

depending on the "quadrant". Thus, there is basically only one "function" of two variables (parameter and argument); the two functions are like "different sheets of that function". The difference between them is one of point of view: that is, which



$\overline{BA} = S_T(\overline{PA};\overline{PB})$ $\overline{AB} = T_S(\overline{PB};\overline{PA})$

Figure 9: Line-to-Line Functions

variable is regarded as the argument (conveniently thought of as small and variable) and which as the parameter (thought of as finite and fixed) for the purpose of investigating the geometry near the intersection. The manner chosen to "traverse from sheet to sheet" * always makes the function differentiable in the argument but not always in the parameter (since the direction of the crossbar becomes ambiguous when it vanishes).

* These comments are not intended to be precise but to give the reader some overall appreciation for the character of these functions.

The Euclidean line-to-line function calculated in the previous chapter (equation 2.5.1) gives specific examples to consider in this context.

$$S_T(\overline{PA};\overline{PB}) = \overline{PA}\sqrt{1-2\frac{\overline{PB}}{\overline{PA}}\cos(\theta_T-\theta_S)+\left(\frac{\overline{PB}}{\overline{PA}}\right)^2}$$

$$T_S(\overline{PB};\overline{PA}) = \overline{PB}\sqrt{1-2\frac{\overline{PA}}{\overline{PB}}\cos(\theta_S-\theta_T)+\left(\frac{\overline{PA}}{\overline{PB}}\right)^2}$$

They are the same except for the effects of the relative signs of the leading factors because, in Euclidean geometry at least, the cosine is a symmetric function of its argument. In effect, these two different functions are two different sets of consistent choices of how to evaluate a square root for various values of the variables.

It is important to note that the line-to-line function is well defined for positive or negative values of either of the variables and that it is unique and consistent for any directions. It is, in effect, a function of its defining directions as well. For instance, if a direction is inverted the result is basically the same function with the sign of the associated argument changed. In the case of the base ray the sign of the function value is reversed as well.

3.3 Derivatives of the Line-to Line Functions

Line-to-line functions are related in various ways because of the tiling of triangles. In

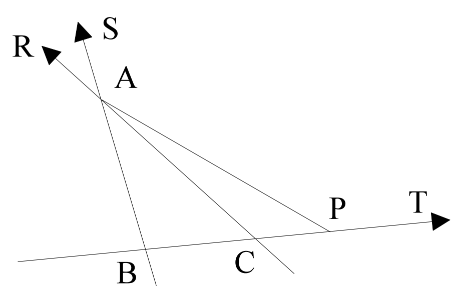


Figure 10: Tiling Relates Derivatives

particular, if they have a common solitary ray one function may be expressed in terms of the other

$$S_T(\overline{BA};\overline{BC}+\overline{CP}) = R_T(\overline{CA};\overline{CP})$$

since segment measure is additive and the value of both line-to-line functions is the same. This leads to a relationship between the derivatives.

$$\frac{d}{d\overline{BP}} S_T(\overline{BA};\overline{BP}) = \frac{d}{d\overline{CP}} S_T(\overline{BA};\overline{BC}+\overline{CP}) = \frac{d}{d\overline{CP}} R_T(\overline{CA};\overline{CP}) \qquad 3.3.1$$

The derivative "at a ray" is independent of the base ray of the function and depends only on the solitary ray.

From the Euclidean expression for the line-to-line function it is easy to calculate its derivative as an example. The result is

$$\frac{dS_T(\overline{PA};\overline{PB})}{d\overline{PB}} = \frac{-\cos(\theta_T-\theta_S)+\frac{\overline{PB}}{\overline{PA}}}{\sqrt{1-2\frac{\overline{PB}}{\overline{PA}}\cos(\theta_T-\theta_S)+\left(\frac{\overline{PB}}{\overline{PA}}\right)^2}}$$

and its behavior can be seen explicitly in this special case. For further reference,

$$\frac{d^2S_T(\overline{PA};\overline{PB})}{d\overline{PB}^2} = \frac{1}{\overline{PA}}\frac{\sin^2(\theta_T-\theta_S)}{\left(1-2\frac{\overline{PB}}{\overline{PA}}\cos(\theta_T-\theta_S)+\left(\frac{\overline{PB}}{\overline{PA}}\right)^2\right)^{3/2}}$$

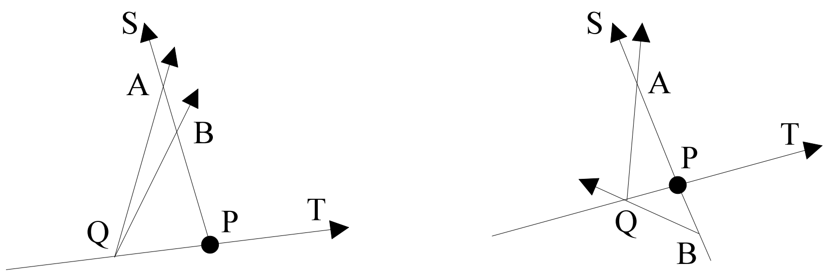
is the second derivative also.

3.4 Angle Derivatives

The original arguments concerning derivatives at an intersection will now be redeveloped specifically for the line-to-line functions for the purpose of basing an important concept on them. Compare a pair of points, A and B, off a line of direction T which are collinear with a point, P, on that same line. The figures below illustrate the two possible situations. Each point corresponds to a different value of the parameter of a line-to-line function.

$$\overline{QA} = S_T(\overline{PA}, \overline{PQ}) \qquad \overline{QB} = S_T(\overline{PB}, \overline{PQ})$$

For the triangle AQB created between crossbars by the common displacement, the salient triangle inequalities are indicated with their corresponding figures. Note that



$$\overline{QA} \le \overline{BA} + \overline{QB} \qquad\qquad \overline{QA} - \overline{QB} \ge \overline{BA}$$

Figure 11: Collinear Points

these inequalities are evaluated in terms of the segment measure for the particular orientations and situations indicated.

The first case is equivalent to an expression,

$$S_T(\overline{PA}, \overline{PQ}) - \overline{PA} \le S_T(\overline{PB}, \overline{PQ}) - \overline{PB},$$

to which the lemma (see page 25) can be applied. The result is

$$\left[\frac{dS_T(\overline{PA}, \overline{PQ})}{d\overline{PQ}}\right]_{\overline{PQ}=0} = \left[\frac{dS_T(\overline{PB}, \overline{PQ})}{d\overline{PQ}}\right]_{\overline{PQ}=0}$$

and
$$\left[\frac{d^2 S_T(\overline{PA}, \overline{PQ})}{d\overline{PQ}^2}\right]_{\overline{PQ}=0} \le \left[\frac{d^2 S_T(\overline{PB}, \overline{PQ})}{d\overline{PQ}^2}\right]_{\overline{PQ}=0}.$$

The second case is similarly equivalent to

$$S_T(\overline{PA}, \overline{PQ}) - \overline{PA} \ge S_T(\overline{PB}, \overline{PQ}) - \overline{PB}$$

and gives similar results.

$$\left[\frac{dS_T(\overline{PA}, \overline{PQ})}{d\overline{PQ}}\right]_{\overline{PQ}=0} = \left[\frac{dS_T(\overline{PB}, \overline{PQ})}{d\overline{PQ}}\right]_{\overline{PQ}=0}$$

$$\left[\frac{d^2 S_T(\overline{PA},\overline{PQ})}{d\overline{PQ}^2}\right]_{\overline{PQ}=0} \geq \left[\frac{d^2 S_T(\overline{PB},\overline{PQ})}{d\overline{PQ}^2}\right]_{\overline{PQ}=0}$$

The "third case", when both the points A and B are below the line is the same as the first case when the direction S is replaced by its inverse direction, S^-, and the points A and B are exchanged. Since, according to previous observation,

$$S^-{}_T(\overline{PA},\overline{PQ}) = -S_T(-\overline{PA},\overline{PQ}) \quad \text{and} \quad S^-{}_T(\overline{PB},\overline{PQ}) = -S_T(-\overline{PB},\overline{PQ})$$

the results of this third case are identical to those of the first.

Theorem 3.4.1. (Angle Derivative). The first derivative of a line-to-line function with respect to its argument, evaluated at zero, is *independent* of the point on the intersecting line (that is, the same for any value of the parameter excluding zero) from which the test segment is dropped.

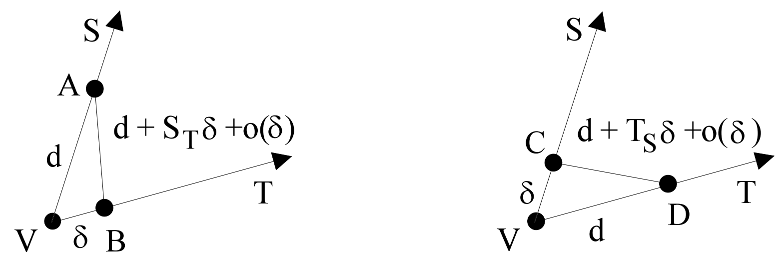
The previous calculations for first and second derivative in the Euclidean case can be evaluated to reveal the values of these quantities to be

$$\left[\frac{dS_T(\overline{PA};\overline{PB})}{d\overline{PB}}\right]_{\overline{PB}=0} = -\cos(\theta_T - \theta_S) \quad \text{and} \quad \left[\frac{d^2 S_T(\overline{PA};\overline{PB})}{d\overline{PB}^2}\right]_{\overline{PB}=0} = \frac{\sin^2(\theta_T - \theta_S)}{\overline{PA}}$$

in this special case. It is instructive to compare these specific results with the general properties that have been deduced.

As in classical geometric terminology, a pair of rays emanating from a common point will be called an *angle*. And the common point will be called the *vertex*. As has been seen, such a geometric structure determines two conceptually distinct (see Figure 12 below) line-to-line functions. The results above reveal that the first derivatives with respect to their arguments at zero are characteristic of the intersection itself and determined only by the two rays involved (provided their orientation is properly taken into account). This is in contrast with other things, like the functions as a whole or the second derivatives, which also depend, in an unknown way, on the point from which the test segment is dropped (that is, on the parameter). It needs to be emphasized that, since these two quantities are conceptually distinct, there is no reason to, a priori, expect them to be related. Furthermore they are, and should be thought of as, functions of both of the *two* component rays of the angle (and not, according to Euclidean habits of thought, as determined by some sort of "relation" between the rays).

These quantities are qualities of their angle. They may be thought of as the linear coefficients in the first order Taylor polynomials of the segment measure from <u>any</u> fixed point on one ray to the line containing the other ray as a function of displacement along that ray from the vertex. They are a sort of derivative of one ray with respect to another. Accordingly it is appropriate and convenient both to adopt



$$S_T = \left[\frac{dS_T(\overline{VA};\overline{VB})}{d\overline{VB}} \right]_{\overline{VB}=0} \qquad T_S = \left[\frac{dT_S(\overline{VD};\overline{VC})}{d\overline{VC}} \right]_{\overline{VC}=0}$$

Figure 12: Two Geometrical Qualities of an Angle

the derivative-like notation* for these quantities illustrated in the figure and to call them *angle derivatives*. Verbally, T_S will be called the derivative of T with respect to S and S_T will be called the derivative of S with respect to T. As previously emphasized: $T_S \neq S_T$ is a possibility. Furthermore, because of the previously noted

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* This is the origin of the notation previously adopted for the point-to-line functions. The functions can always be easily distinguished by context from the derivatives. This possibility of confusion is preferable to the alternative of a needless proliferation of notation for variations on a theme.

symmetry properties of the line-to-line functions, these quantities are "anti-symmetric" in both rays of the angle; that is, inversion of either ray's orientation reverses their sign.

The angle derivative can be explicitly calculated in the special Euclidean case.

$$S_T = -\cos(\theta_T - \theta_S) \quad \text{and} \quad T_S = -\cos(\theta_S - \theta_T).$$

This connection of the angle derivatives with a "cosine" will persist, albeit in a generalized form, in the unspecialized case. In this special case the two angle derivatives are equal because the circular cosine is symmetric in its arguments. This feature does not survive generalization.

3.5 Second Derivatives

The conclusions that can be drawn on the basis of the results of the previous section concerning the second derivative of a line-to-line function with respect to the argument (at zero) are less conclusive than those concerning the first derivatives. On the side of the solitary ray indicated by the base ray they tend to increase as the intersection is approached (more precisely they are non-decreasing as the parameter approaches zero). On the other side they tend to decrease (are non-increasing) as the intersection is approached. Furthermore, the minimum on the base ray side must be greater than the maximum on the other side.

These results are consistent with the second derivative having the same sign as the parameter, never vanishing and diverging in magnitude as the parameter becomes small as in the Euclidean example above. However, such conclusions are unwarranted without additional results (for example, that the magnitude of the second derivative tends to zero with large parameters; that is, far away from the intersection). It is possible, for instance, that some such second derivative might vanish identically at some line. And stranger possibilities still remain. For instance, that somewhere on one side the second derivative might actually go thru zero, reverse sign and tend to a non-zero magnitude for large values of the parameter.

Nevertheless, there are a few additional things that can be said. Tho the following theorem seems like a pretty meager result, surprisingly, it is useful.

Theorem 3.5.1. (General Sign Matching). For any parameter in any line-to-line function there is a value of the argument at which the second derivative has the same sign as the parameter.

Proof. Given a line of direction *T* and a point *A* not on it, consider the length of the segment between *A* and an arbitrary point *Q* on the line. As *Q* moves off toward either end of the line then the displacement of *Q* from any reference point, P, on the line increases monotonically (because the points are uniquely measured by their additive distances on the line). By the triangle inequality $|\overline{PQ}| \leq |\overline{PA}| + |\overline{QA}|$ and so

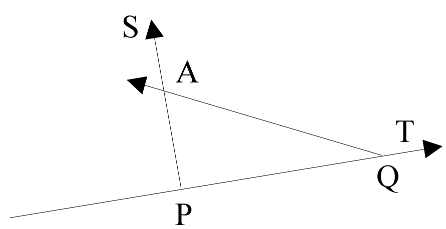

Figure 13: Asymptotic Behavior

$|\overline{QA}|$ must eventually increase arbitrarily in magnitude. In particular, for *Q* sufficiently to the left of *P* or to its right $|\overline{QA}|$ must eventually exceed $|\overline{PA}|$. For $\overline{PA} > 0$ this means that the first derivative of the line-to-line function corresponding to this situation must be negative at some point to the left of *P* and must be positive at some point to the right of *P*. For this to be possible, somewhere in-between these points the second derivative must be positive. The corresponding argument for $\overline{PA} < 0$ shows that there must be some point where the second derivative is negative.

3.6 Convex and Analytic Physical Geometries

The properties of the second derivatives of the line-to-line functions are crucial to the character of the geometry. The following definitions name some interesting special situations.

Definition. A physical geometry in which the derivatives of the line-to-line functions (with respect to the argument) are monotonic functions (that is, they only change in one direction) of the argument will be called *convex**.

The term is motivated by the concept of a convex function. The connection is brought out by the theorems below.

Definition. A physical geometry in which the functions to which the continuity axiom applies are analytic will be called an *analytic* physical geometry. In particular, this means that those functions have derivatives of all orders.

Theorem 3.6.1. (Convex Sign Matching). In the convex case the second derivatives of the line-to-line functions take the sign of the parameter. If they vanish at any value of the parameter they must vanish for all larger values of the parameter and if the geometry is analytic they must vanish for all values of the parameter.
Proof. For a given parameter the sign of the second derivative cannot change with the argument. Perhaps it might go to zero but it can never go thru zero; otherwise the direction of change of the first derivative would reverse and it would not be monotonic. Then it cannot have a sign opposite to that of the parameter for any value of the argument since, by the General Sign Matching Theorem, it must have a matching sign for some value of the argument. Furthermore, if it were actually to

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* Dr. Plaut tells me that this notion is closely related to the idea of "non-positive curvature" and suggests Ballmann as a reference.

vanish for some value of the parameter then since for values of the parameter of larger magnitude it cannot be larger and cannot be negative it must also vanish. For the analytic case vanishing on such an interval requires that, as a function of the parameter, it be identically zero.

The following theorem applies everywhere in a convex geometry. But it also applies generally to any case where the second derivative of a line-to-line function does not change sign.

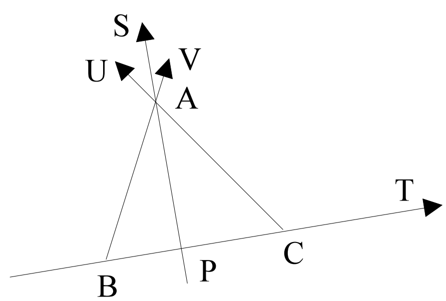

Figure 14: Estimation of Line-to-Line Functions

Theorem 3.6.2. (Interpolation Estimation). Suppose, with reference to the figure above, that there is a finite interval, BC, about P on which the second derivative,

$$\frac{d^2}{dx^2} S_T(\overline{PA}, x) \text{ for } \overline{PB} \le x \le \overline{PC},$$

and has the same sign. Then

$$\left.\begin{matrix} \overline{BA} + V_T \overline{BQ} \\ \overline{CA} + U_T \overline{CQ} \end{matrix}\right\} \le S_T(\overline{PA}, \overline{PQ}) \le \frac{\overline{BQ}\,\overline{CA} + \overline{BA}\,\overline{QC}}{\overline{BC}} \quad \text{for + sign}$$

and

$$\frac{\overline{BQ}\,\overline{CA} + \overline{BA}\,\overline{QC}}{\overline{BC}} \le S_T(\overline{PA}, \overline{PQ}) \le \left\{\begin{matrix} \overline{BA} + V_T \overline{BQ} \\ \overline{CA} + U_T \overline{CQ} \end{matrix}\right. \quad \text{for } - \text{ sign}$$

where Q is a point between B and C.

This is just a standard convex function result. Unfortunately these estimates seem to be so weak that they are of little use.

Theorem 3.6.3. In an analytic, convex geometry two lines which intersect a third with the same angle derivative cannot, themselves, intersect. That is, they are parallels.

Proof. Suppose they did intersect. There is a point-to-line function determined by the point of intersection and the "third line". Since its derivatives are monotone they must be constant on the interval between the two lines in question. The point-to-line function must, therefore, be linear. This is impossible so the two lines cannot intersect.

Both convex and analytic geometries include Euclidean geometry but are a wider categories. These interesting types of physical geometry worthy of further investigation; especially with regard to how closely they are related to Euclidean geometry. But, theorems such as the above suggest they may be overly specific for more general interest.

3.7 Degeneracy

Several applications of the triangle inequality to the defining triangle of a line-to-line

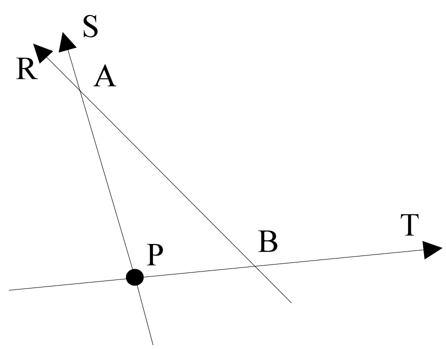

Figure 15: Line-to-Line Functions and the Triangle Inequality

function require that $\left\| \overline{BA} \right| - \left| \overline{PA} \right\| \leq \left| \overline{PB} \right|$. Since $\overline{BA} = S_T(\overline{PA}; \overline{PB})$ has, by definition, the same sign as $\overline{PA} = S_T(\overline{PA}; 0)$ this means that

$$\left|\frac{S_T(\overline{PA};\overline{PB})-S_T(\overline{PA};0)}{\overline{PB}}\right|\leq 1$$

and so by taking the limit:

$$|S_T|=\left|\left[\frac{d}{d\overline{PB}}S_T(\overline{PA};\overline{PB})\right]_{\overline{PB}=0}\right|\leq 1\,.$$

For $S=T$ and $S=T^{-}$ this derivative easy to evaluate. It is -1 and $+1$, respectively, so the extremes are realized. Furthermore, in view of the tiling relation (3.3.1) between the derivatives of line-to-line functions with a common solitary ray:

$$\left|\frac{d}{d\overline{PB}}S_T(\overline{PA};\overline{PB})\right|\leq 1$$

more generally.

The triangle inequality is a strict one unless the points are collinear. Therefore, for A not on the head or the tail of T ($S\neq T,T^{-}$), one might expect that the derivative is strictly less than magnitude one. And this would suggest that this quantity might be used to measure the relative direction of lines (as will eventually be done). However, since the limit of quantities bounded away from 1 may in fact be one, the above argument does not justify this expectation. This possibility requires the following definition

Definition. A physical geometry in which there exist two distinct intersecting directions *S* and *T* such that $S_T=+1$ or $S_T=-1$ is said to be *degenerate*. Otherwise it is *non-degenerate*.

Euclidean geometry is non-degenerate. Whether or not degeneracy is possible is unknown. So far, no proof has been found to exclude it and no example has been found to demonstrate it. For the present, therefore, it is necessary to allow for the possibility.

However, a physical geometry that was degenerate would have some rather peculiar properties. If, say, $S_T = +1$ then $d/dx\, S_T(p,x)$ cannot be 1 in any finite interval about $x = 0$. Otherwise one could integrate over that interval and find that the sides of a finite triangle added as if the vertices were collinear in violation of the triangle inequality. Therefore, since $+1$ is the maximum value it can take:

$$\frac{d}{dx} S_T(p,x) < 1 \text{ for } x \neq 0$$

in some neighborhood of 0. The value of the derivative at the point $x = 0$ is thus an absolute maximum. This means, for instance, that convex geometries cannot be degenerate since this is inconsistent with monotonicity. Furthermore, the second derivative,

$$\left[\frac{d^2}{dx^2} S_T(p,x) \right]_{x=0} = 0,$$

must vanish for <u>all</u> values of the parameter!

The chapter on direction (Chapter 5) will mainly be concerned with non-degenerate geometries. But, along the way, additional weird properties of degenerate geometry will be obtained. These will, for instance, lead to the conclusion that analytic geometries cannot be degenerate either.