Introduction

What is the essential nature of physical space? This is a physical question but the answer sought is a mathematical one -- a mathematical description faithful to reality. The classical geometry of the ancient Greeks is at least such an excellent approximation that it surely must contain something of the answer to this question. But what about it is central and universal and what is inessential and particular? How can the physically appropriate generalization be identified?

1.1 Generalization

The discovery, during the Renaissance, that geometry could be modeled in algebraic terms gave rise to analytic geometry. This provided the basis on which Riemann developed a differential generalization of geometry. But analytic geometry introduces artifacts which pertain to the model and are extraneous to the geometry itself. It is, as a result, characteristically difficult, in this form of geometry, to separate out the essential from the extraneous. The question then arises as to the nature of the Riemannian generalization. Is it purely a generalization of the geometry or is it, at least in part, a generalization of inessential aspects^{*} of the particular model? Is it inadvertently specialized[†] in an essential way (at least for use as a description of physical space) by the peculiarities of the algebraic model or its Euclidean basis? Such imponderable questions cannot, of course, be answered definitively. However, it might be possible to demonstrate other paths leading to alternative generalizations.

^{*} Such as, perhaps, a quadratic differential form.

[†] For example, when an algebraic parameter is set to zero its term may vanish. From the expression that remains it is impossible to guess that there was a term, let alone its form. This is just an illustration showing that the possibility raised is credible not a proof that it is a reality or if it is a reality that it is important.

Chapter 1

The possibilities raised by these questions suggest that it might be worthwhile, therefore, to look for non-algebraic methods of generalization and to investigate them. Even if they do not lead to anything new, that, in itself, is interesting and a possible source of new insight into geometry. However, if novel geometric possibilities result not only would they be interesting in themselves but their utility for physical theory should be evaluated.

Accordingly, it might be interesting to develop a differential generalization rather in the spirit of the Greek approach to geometry. The results obtained directly from this pure, so-called synthetic, geometry could then be compared with the Riemannian version upon which the theory of gravitation has hitherto been based. This is the program being undertaken.

1.2 Physical Motivation

Much has been made of the contrast between the so-called parallel postulate in Euclid's geometry and the rest of his axioms. While the other axioms seem to be nearly trivial statements of existence and continuity, only the parallel postulate, whose independence is notorious, appears to have substantial content. Actually, in Euclid's geometry, there are also implicit assumptions of isotropy and uniformity that have similar substance. Such assumptions are made explicit in modern treatments of geometry; by, for example, taking SAS congruence of triangles as an axiom. In any case, there is an evident contrast between the substantial and the other axioms. This difference in the character of the two types of axioms is attested to by the fact that for millennia isotropy and uniformity were taken for granted and the parallel postulate was suspected to be a theorem. How is this contrast to be understood?

Perhaps this contrast might be understood as that between a particular physical condition of space (the substantial axioms) and the more spare and essential nature of space-in-itself that is universal. This leads to the thought that exploring the general

consequences of axioms of the latter sort, by themselves, might yield an interesting geometry and provide a promising basis for a differential generalization. Since these kinds of axioms *seem* to have so little content it will perhaps be surprising that the resulting theory, far from being vacuous or stillborn, is, even to the limited extent to which it is developed in the following, geometrically fruitful.

Once this mathematical theory is understood it may be possible to identify the range of geometrical possibilities remaining within it. It is hoped that, then, by interpreting these possibilities as the physical condition of space, a further understanding of gravitation may be achieved. However, in order to make it possible to decide whether or not there are such possibilities for physical content it is first necessary to explore what is basically entirely mathematics. After this mathematical preliminary is substantially completed, and only then, it may be possible to relate the results to physics.

1.3 Scope of Paper

This thesis is just the introductory part of a paper on the work that has been done so far in carrying out this limited program. It is begins by establishing clear, simple and minimal physically intuitive axioms in the following chapter and then continues by working out some of their consequences. It ends at the point where the study of the tangent space begins.