

**Differential Synthetic Geometry:
a Possible Foundation for a Theory of Gravitation**

by
Craig Spencer

Submitted in Partial Fulfillment
of the
Requirements for the Degree
Master of Science

Supervised by
Professor Conrad Plaut
Department of Mathematics, University of Tennessee
and
Professor Yongli Gao
Department of Physics and Astronomy
The College
Arts and Sciences

University of Rochester
Rochester, New York

1996

Power may obstruct knowledge for a time but knowledge will prevail.... Even under oppression scholars can take heart with the victorious knowledge ... ‘and still it moves!’
-- Max von Laue

The scientific incontrovertibility of physics leads directly to the ethical demand for veracity and honesty. And justice is inseparable from truth... Just as the laws of nature work consistently and without exception, in great things and in small, so too people cannot live together without equal justice for all.
-- Max Planck

Science makes people reach ... for truth and objectivity; it teaches people to accept reality, with wonder and admiration, not to mention the deep joy and awe that the natural order of things brings to the true scientist.
-- Lise Meitner

Dedicated
to
Genuine, Honest Scientists

Curriculum Vitae

The author was born in Salem, Oregon on January 19, 1946. He attended Oregon State University as a Special Student for the 1963-64 academic year. He did his baccalaureate work at the California Institute of Technology from 1964 to 1968 receiving his Bachelor of Science degree (with honor) in the field of Physics in 1968. He attended Montana State University for the Fall of 1973. He came to the University of Rochester in September 1988 with vain hopes of doing post-graduate work.

Acknowledgments

I owe a great deal to three particular people who made this thesis possible. They have my profound thanks.

Tho I have been thinking about this material for a long time I never made significant progress until I took Professor Sanford Segal's course in Non-Euclidean Geometry in the Fall of 1991. It revealed to me the efficacy of the methods of synthetic geometry beyond the high school level. This body of mathematics is, at great loss, neglected in the physics curriculum (and, astoundingly, by mathematicians as well).

Without the advice, help and friendship of Yongli Gao I would not have been able to submit this thesis. I owe him a great deal.

Conrad Plaut was the only person who was willing to seriously look at any of my paper. Otherwise it would have been entirely buried by calculated default. He has been unaccountably generous with his time in serving as thesis advisor. Without his willingness to do so the Physics Department would not have formed of the required thesis committee. He has thereby made possible the submission of this thesis. And his knowledge, advice and help have improved it considerably.

Abstract

An interest in developing a differential version of synthetic geometry is motivated by its potential for gravitational theory, especially a version which might gracefully incorporate spin. This thesis only addresses the needed mathematical foundations; physical applications, if there are any, will have to come later. An axiom set for two dimensions is proposed, some of its consequences explored and, to a certain point, considerable progress is made.

Table of Contents

Chapter	1	Introduction	1
	1.1	Generalization	1
	1.2	Physical Motivation.....	2
	1.3	Scope of Paper.....	3
Chapter	2	Synthetic Geometry	4
	2.1	Incidence Geometry	4
	2.2	Metric Geometry.....	5
	2.3	Elementary Concepts.....	9
	2.4	Oriented Measure	12
	2.5	Continuity.....	15
	2.6	The Geodesic Hypothesis	17
	2.7	Euclidean vs. Physical Geometry	19
	2.8	Busemann's Examples.....	19
	2.9	A Natural Example	20
	2.10	Global and Local Differential Geometry.....	22
	2.11	Next.....	23
Chapter	3	Fundamentals	24
	3.1	Derivative Comparisons	24
	3.2	Line-to-Line Functions	27
	3.3	Derivatives of the Line-to-Line Functions.....	29
	3.4	Angle Derivatives	30
	3.5	Second Derivatives	34
	3.6	Convex and Analytic Physical Geometries.....	36
	3.7	Degeneracy	38

Chapter	4	Two Dimensions	41
	4.1	Pasch's Postulate	41
	4.2	Plane Geometry	42
	4.3	Expectations	44
Chapter	5	Direction	45
	5.1	Order Inequalities	45
	5.2	The Degenerate Case	50
	5.3	The Non-Degenerate Case	52
	5.4	Distinguishing Rays	53
	5.5	Measuring Direction.....	54
	5.6	Perpendiculars	56
Summary		61
Appendix A		Models and Examples	62
	A.1	The Continuity Axiom.....	63
	A.2	G-Space	64
	A.3	Busemann's Examples.....	65
	A.4	Busemann Continuity	67
	A.5	The Busemann Triangle Inequality.....	69
	A.6	Busemann's Specific Cases.....	70
	A.7	Isotropic-Scaling Geometry	71
Bibliography		74

List of Figures

Figure 1	Rays and Directions	12
Figure 2	Alternate Paths	13
Figure 3	Segment Measure and Orientation	14
Figure 4	Point-to-Line Function	16
Figure 5	Distance from a Point to a Line	24
Figure 6	Collinear Points on One Side of a Line	24
Figure 7	Collinear Points on Opposite Sides of an Intersecting Line.....	26
Figure 8	“Supplementary and Vertical Angles”	27
Figure 9	Line-to-Line Functions	28
Figure 10	Tiling Relates Derivatives	29
Figure 11	Collinear Points	31
Figure 12	Two Geometrical Qualities of an Angle	33
Figure 13	Asymptotic Behavior.....	35
Figure 14	Estimation of Line-to-Line Functions.....	37
Figure 15	Line-to-Line Functions and the Triangle Inequality	38
Figure 16	Mutually Monotonic Crossbars	44
Figure 17	Order Inequalities	46

Figure 18	Other Order Inequalities	49
Figure 19	Continuous Transition.....	51
Figure 20	Degenerate Angles.....	52
Figure 21	Continuity of Angle Derivatives #1	54
Figure 22	Continuity of Angle Derivatives #2.....	55
Figure 23	Perpendicular Erected to a Line at a Point.....	56
Figure 24	Point and Line	57
Figure 25	Perpendicular Dropped from a Point to a Line	57
Figure 26	Near a Perpendicular Dropped from a Point.....	58
Figure 27	Perpendiculars of Perpendiculars	59
Figure 28	Twist Triangle	60
Figure 29	Using a Euclidean Correspondence	64
Figure 30	Constructing Busemann's Metric	66
Figure 31	Examples of Projecting Three Points.....	70